

INTRODUCTION TO LINEAR ALGEBRA

Fifth Edition

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Introduction to Linear Algebra, 5th Edition

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The website for this book is math.mit.edu/linearalgebra.

The Solution Manual can be printed from that website.

Course material including syllabus and exams and also videotaped lectures are available on the book website and the teaching website: web.mit.edu/18.06

Linear Algebra is included in MIT's OpenCourseWare site ocw.mit.edu.

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The front cover captures a central idea of linear algebra.

$Ax = b$ is solvable when b is in the (red) column space of A .

One particular solution y is in the (yellow) row space: $Ay = b$.

Add any vector z from the (green) nullspace of A : $Az = 0$.

The complete solution is $x = y + z$. Then $Ax = Ay + Az = b$.

The cover design was the inspiration of Lois Sellers and Gail Corbett.

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